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The Idiosyncratic Volatility Puzzle

Eduardo Cortellesi

Abstract

Monthly sorting stocks into quintiles based on idiosyncratic volatility levels we find a negative relationship between idiosyncratic volatility and returns, providing additional evidence of the Idiosyncratic Volatility Puzzle after the crisis in the American equity market. We test several holding periods (1, 3, 6 and 12 months) finding that the relation between idiosyncratic volatility and returns is holding-period dependent, because it presents an inverted U-shape trend. A wavelet multi-resolution analysis performed on our data shows the contribution of different frequencies to the Puzzle, reporting the relevance of heterogeneity of investors' investment horizon hypothesis. Several performance evaluation measures are then computed for two trading strategies exploiting the Puzzle.

1. Introduction

We examine the relationship between idiosyncratic volatility and cross-section of returns, aiming to verify if the negative relationship found by Ang et al. (2006)[1] is still present in the post-crisis period (2010-2018). Hence, for each firm in the American equity market, we construct from the individual stocks 5 portfolios with different levels of idiosyncratic risk. We measure then, the performance difference between the portfolio made of stocks with highest level of idiosyncratic risk and the portfolio based on stocks with the lowest level of idiosyncratic risk. Afterwards a trading strategy exploiting the patterns we find in the data is tested with several evaluation performance measures. We then analyse if the relationship between idiosyncratic volatility and returns is holding-period dependent, in order to test if heterogeneity of investors' investment horizon hypothesis is verified by our findings. To further test this recent hypothesis in literature, we apply the wavelet transform to study the contribution of each frequency in our data to the Puzzle.

Our *novel contribution* is to use Ang et al. (2006)'s approach to examine the idiosyncratic volatility-return relationship in a different sample period[2] and to evaluate the performance of two trading strategies exploiting the Puzzle. In addition, we analyse if the relationship is holding-period dependent using a strategy with increasing holding periods (1, 3, 6 and 12 months). To relate the inverted U-shaped relation we observe, to the heterogeneity of investors' investment horizons hypothesis, we perform a time-frequency analysis with the wavelet transform on the American equity market post-crisis.

In literature, the theme of this thesis is known as "The Volatility Puzzle". We focus on the idiosyncratic side of the volatility and following other papers' example, from now on we refer to it as the "IVOL Puzzle". Studying the IVOL Puzzle can be helpful both from a factor investing point of view (as a trading strategy) and as a stronger theoretical framework for all the investors which fail to diversify.

We use a dataset from Wharton Research Data Service, composed by the daily returns of the stocks belonging to the same exchanges Ang et al. (2006) used, for the post-crisis period. We monthly construct the 5 portfolios formed on IVOLs using the trading strategy 1/0/1. We find evidence of a negative relationship between IVOL and returns because the portfolio 1 (formed on low volatility stocks) outperforms the portfolio 5 (based on high volatility stocks). The fact the alphas are substantial (-0.72% relative to CAPM and -0.59% relative from Fama-French

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[1] Their sample period is 1963-2000.

[2] In the American equity market, as for Ang et al. (2006).

three-factor model) and statistically significant brings additional evidence of the existence of the IVOL Puzzle after the crisis.

Given our findings, we compute performance evaluation measures of four well-known strategies based on: Market, Size, Value and Momentum factors. We compare these strategies to the strategy of going long on low idiosyncratic volatility stocks and short on high idiosyncratic volatility stocks. We find our strategy and the momentum strategy performing well relative to the market index. Afterwards we quantify the cost for a mean-variance optimizer investor, with an indexed position, of ignoring low idiosyncratic volatility stocks. By tilting its position towards the low idiosyncratic volatility stocks, it increases substantially its utility function.

The trading strategy 1/0/3, 1/0/6 and 1/0/12 bring other relevant discoveries. The trading strategy 1/0/3 shows the same patterns of the trading strategy 1/0/1, where portfolio 5 underperforms the portfolio 1. The alphas are relevant and statically significant, meaning that the CAPM and FF3 models still fail to price the portfolios. The trading strategy 1/0/6 shows the absence of a difference in performance both in returns and alphas between portfolio 1 and 5. The trading strategy 1/0/12 displays again the Puzzle's existence with the same patterns of the strategy 1/0/1 and 1/0/3.

Malagon et al. (2015); Yin et al. (2019) use the Wavelet Multi-Resolution Analysis to separate investor classes and decompose a time series into different time horizons. In our case instead of decomposing the time series, playing with the setting of the L/M/N strategy we test different holding periods, separating the investors from active (frequent rebalancing) to more passive investors (yearly rebalance). Anyway, time scales determine the overall return we capture, therefore different compensations required by investors with different time horizons affects the compensation of the overall holding periods. The sum of the compensations required for all the time scales inside an holding period makes the final compensation we observe, hence a change in compensation for increasing holding periods implies a compensation for bearing idiosyncratic risk that is investment-horizon dependent.

Finally, we test the hetereogeneity of investors' investment horizons hypothesis. Following the framework of Malagon et al. (2015) we apply the wavelet multi-resolution analysis, decomposing our data into 7 different frequencies representing the behaviour of different kind of investors. The analysis reports a negative relationship between firm-specific risk and returns for short term investors (investment horizon from 2 to 32 days), a positive one for medium term investors (32 to 128 days) and a negative one for longer investment horizons (> 128 days).

2. Literature on the Idiosyncratic Volatility Puzzle

2.1 Preamble

By theory, there should be a premium to compensate investors for holding assets that are not diversified. Diversification smooths out the firm specific risk by holding an enough large number of assets. The consequence of diversification is a lower risk faced, hence a better return for unit of risk in our portfolio. The reason why facing less risk means a better mean-variance optimization, is that under certain general assumptions the idiosyncratic risk is not priced (compensated) as the systematic risk. If investors were rational individuals, they should not face idiosyncratic risk and it should not even be priced. Given the empirical evidence, investors fail to fully diversify their investments, therefore an investigation on how idiosyncratic risk affects portfolio's performance was needed both for theoretical and investment purposes. Several papers over the years, investigating how idiosyncratic risk is priced by the market, have found mixed evidence. In academic literature the evidence about Idiosyncratic Volatility Puzzle is mixed: there are researchers that found a significant positive relationship between idiosyncratic volatility and average returns as Fu (2009), there are others which failed to find a significant relationship between these two variables as Bali and Cakici (2008) and finally there is also evidence of a negative relationship as Ang et al. (2006).

2.2 Hypotheses

The literature's debate over the Idiosyncratic Volatility Puzzle had two main periods. From 2006 till 2009, the main effort has been to weaken the robustness of Ang et al. (2006)'s findings. Even if some critiques may have been economically reasonable and empirically proved, see Fu (2009) about the time-varying volatility's nature, statistically significant evidence of the Puzzle was still unexplained. Ang et al. (2009) have brought huge consensus about the robustness of their results, in fact after 2009 more possible explanations to the Puzzle came up. The most famous one are: heterogeneity of investors' investment horizons, lottery preference (behavioural explanation), market frictions, average variance beta (Chen and Petkova (2012)) and IVOL as an information content (Jiang et al. (2009)). Between all the hypotheses which have been used as reason behind the Puzzle, heterogeneity of investors' investment horizons captures our attention because it can potentially explain both the significant negative relationship between IVOL and returns and the mixed evidence found in literature. This hypothesis states that the compensation investors demand for bearing idiosyncratic risk could be horizon dependent. Malagon et al. (2015), applying Wavelet Multi-Resolution Analysis to disentangle the different time horizons, find a negative relationship between IVOL and returns for the short term investors while the relationship gets positive for long-term investors. Yin et al. (2019) find the Puzzle for short-term investors, a positive relationship for middle-term investors and a negative relationship again for long term investors.

Aim of the thesis is to search for evidence about IVOL Puzzle, considering the post-crisis sample period (2010-2018). Given the evidence found in the literature, applying the same framework of Ang et al. (2006, 2009) we expect to find supporting evidence to the Puzzle, since we fail to notice reasons why the compensation demanded by investors exposed to idiosyncratic risk should have been changed[3].

About the heterogeneity of investors' investment horizons hypothesis, we are going to test different holding period (different values for N) of the trading strategy L/M/N, to check if there is a change in the compensation required by investors bearing the idiosycnratic risk for a holding period of 1,3,6 and 12 months. Given the Malagon et al. (2015); Yin et al. (2019)'s results, we expect to observe supporting evidence to the heterogeneity of investors' investment horizons hypothesis.

2.3 The Idiosyncratic Volatility

We defined volatility of an asset as the standard deviation of returns with a given frequency, therefore it can be easily measured. On the other hand, idiosyncratic volatility can only be estimated from the model's residuals, therefore is model dependent. If idiosyncratic volatility is model dependent then its accuracy is model dependent too, hence the better the model the better the idiosyncratic volatility we estimate.

Risk, intended as the standard deviation of returns over time, can be divided into two main components. When a risk is faced by all the securities in the market and can't be diversified because related to macroeconomic factors, is classified as systematic risk. As systematic are considered: the interest rate risk, the market risk, the purchasing power risk, the exchange rate risk and the political risk[4]. On the other hand, the idiosyncratic risk is an industry/firm/stock specific risk which can be diversified away just increasing the number of stocks inside the portfolio. Campbell et al. (2001) stated that "the number of randomly selected stocks needed to achieve relatively complete portfolio diversification" is about 50. The intuition is that usually assets are not perfectly correlated, therefore an additional asset will decrease the portfolio's idiosyncratic risk. Using the Ang et al. (2006)'s methodology, the IVOL computation derives

[3] Sample period of Ang et al. (2006, 2009) is 1963-2000.
[4] https://efinancemanagement.com/investment-decisions/systematic-risk

from the squared root of the residuals' variance $\sqrt{(Var(\epsilon\_i\ ,\_t\ )\ )}$ from the Fama-French 3-factors model (OLS multivariate regression):

$$r_{it} = \alpha_i + \beta_{i,mkt}MKT_t + \beta_{i,SMB}SMB_t + \beta_{i,HML}HML_t + \epsilon_{i,t} \tag{1}$$

Therefore, from now on when we talk about IVOL we refer to idiosyncratic volatility relative to the Fama-French three-factor model.

3. Trading strategy

Following Ang et al. (2006), we define our framework as the trading strategy L/M/N. At a point in time t we sort the daily stocks returns based on the L-months lagged IVOLs into 5 quantiles, then we wait M-months and eventually we hold these portfolios (the 5 quantiles) for N-months[5]. The IVOLs are constructed monthly over daily returns. We examine if going short on P5 and long on P1 is profitable. We analyse the following trading strategy's settings: 2/0/1 with monthly rebalancing, 1/0/1 with monthly rebalancing, 1/0/3 with quarterly rebalancing, 1/0/6 with semesterly rebalancing and 1/0/12 with annual rebalancing.
The difference in an investor who does monthly rebalancing based on a factor (IVOL in our case) compared to a yearly rebalancing is the different level of activeness used to manage his portfolio. Investors who decide to rebalance every year are closer to a passive investing management while monthly rebalancing investors are more active in their portfolio management.

3.1 Dataset from CRSP

The analysis now shifts over a dataset from Wharton Research Data Service[6]. It contains the daily returns of stocks on primary listings for NYSE, NYSE MKT (previously known as AMEX), NASDAQ and ARCA exchanges. The time period considered is the post-crisis one, therefore 2010-2018. Over the dataset will be tested the following trading strategies: 1/0/1, 1/0/3, 1/0/6 and 1/0/12. The columns/variables it contains are: daily returns, price per share and number of share outstanding. The last two variables are needed to compute the value-weighted returns of each portfolio. Multiplying them, we obtain the market capitalization which will be used to weight the returns inside the portfolios.

[5] The portfolios returns at the end of the M period are value-weighted
[6] http://www.crsp.com/products/research-products/crsp-us-stock-databases

3.2 Trading strategy 1/0/1

We construct monthly portfolios of stock returns based on five levels of the 1 month lagged IVOLs. The results are reported in a table which is in the layout similar to Ang et al. (2006)'s table for comparison purposes. This means the statistics computed for the 5 portfolios are: mean, standard deviation, market share (intended as average market capitalization of the portfolio over the sum of the 5 portfolios' market capitalizations) and alphas from CAPM and Fama-French three-factor models. In order to be as close as possible to the real application of a trading strategy, to compute the value-weighted returns we used as weights the market capitalization of the first day of the month considered. Results are in Table 1.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	0.94	2.88	0.21	0.41***	0.38***
				[3.72]	[3.70]
P2	0.99	3.66	0.40	0.29***	-0.26***
				[2.63]	[2.89]
P3	0.99	4.30	0.25	0.18*	0.18*
				[1.81]	[1.68]
P4	0.81	4.82	0.11	-0.06	-0.01
				[-0.40]	[-0.06]
P5	0.63	5.66	0.04	-0.31	-0.22
				[-1.22]	[-1.00]
P5-P1	-0.32			-0.72**	-0.59**
	[-0.80]			[-2.53]	[-2.50]

Table 1: Forming value-weighted quintile portfolios every month we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in monthly returns between portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) *t*-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/1.

Our findings have the same patterns of Ang et al. (2006). In Table 1 P5-P1 return is -0.32% but it's not statistically significant. The CAPM and Fama-French three-factor model's alphas are in magnitude smaller than Ang et al. (2006), but are statistically significant. Overall, we observe additional evidence of the IVOL Puzzle, since CAPM and Fama-French three-factor model misprice the P5-P1 portfolio's alphas yielding statistically significant monthly alphas of -0.72% and -0.59% on the long P5 short P1 strategy. The decreasing pattern in market share from P1 to P5 is decreasing starting from P3.

3.3 Trading strategy 1/0/6

This time the holding period of the portfolios sorted based of 1 month lagged IVOL is 6 months. The rebalancing is semesterly. Results are in Table 2.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	5.63	7.22	0.25	1.83***	0.68
				[4.27]	[1.46]
P2	7.10	8.00	0.20	2.89***	2.74***
				[15.57]	[4.42]
P3	5.64	8.86	0.18	1.09	0.88
				[1.44]	[0.86]
P4	5.55	9.39	0.18	0.72	0.13
				[0.68]	[0.17]
P5	5.42	9.00	0.19	0.88	0.96
				[0.90]	[0.52]
P5-P1	-0.21			-0.95	0.28
	[-0.40]			[-1.12]	[0.25]

Table 2: Forming value-weighted quintile portfolios every six months we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in semesterly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in semesterly returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French threefactor models. Robust Newey and West (1986) *t*-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/6.

Compared to 1/0/3, the gap in return between P5 and P1 substantially shrinks (-0.21% for 1/0/6 compared to -1.30% for 1/0/3). Since the holding period is twice the size, assuming the compensation was still negative from month 3 to 6 and given the same pattern we found with previous strategies, we were expecting a bigger gap.

The fact IVOL Puzzle vanished with a holding period of 6 months could be caused by several reasons. We state that testing different holding periods is a way to bring new evidence to the heterogeneity of investors' investment horizon hypothesis. The fact the IVOL Puzzle (a lower compensation for high IVOL stocks compared to low IVOL stocks) is reduced till to disappeared, can be explained by the presence of a positive compensation for bearing idiosyncratic risk approximately from the 3rd to the 6th month.

3.4 Trading strategy 1/0/12

Holding period is set to 12 months for portfolios based on 1 month lagged IVOL. Results are in Table 3.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	10.60	9.53	0.34	5.32	5.26**
				[0.61]	[1.99]
P2	10.19	13.26	0.35	2.75***	2.01***
				[4.40]	[6.96]
P3	9.78	15.31	0.20	1.41	1.04
				[1.14]	[1.14]
P4	6.63	15.91	0.09	-1.99**	-2.54***
				[-2.29]	[-7.24]
P5	6.17	22.57	0.03	-5.34***	-3.75***
				[-5.44]	[-5.34]
P5-P1	-4.44			-10.66***	-9.00***
	[-1.34]			[-4.63]	[-4.17]

Table 3: Forming value-weighted quintile portfolios every twelve months we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in annual percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in annual returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) *t*-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/12.

Increasing additionally the holding period from 6 to 12 months, the IVOL Puzzle appears again. Standard Deviations increase from P1 to P5 while Market Share decrease from P1 to P5. The usual patterns from P1 to P5 for each statistics are back. The alphas' *t*-statistics are more robust than the other strategies. The 1/0/12 strategy brings again evidence of the IVOL Puzzle.

4. Trading strategy's results

4.1 Idiosyncratic Volatility Puzzle's post-crisis evidence

Main theme of the thesis is to search for evidence about the IVOL Puzzle in the post-crisis period. We apply several trading strategies (2/0/1, 1/0/1, 1/0/3, 1/0/6 and 1/0/12) based on IVOL, to see if the negative relationship between IVOL and returns is still present after the crisis. Across these strategies, we always have found relevant and strongly significant spread alphas between P5 and P1 relative to CAPM and Fama-French 3-factor models (but for trading strategy 1/0/6). We find relevant and statically significant alphas (but for trading strategy 1/0/6), meaning the strategies are able to ”beat the market”. This expression is used when active managers form portfolios capable of gaining actual returns that exceed risk-adjusted expected returns. The total actual return minus the risk-adjusted expected return equals the ”alpha” gained and it measures the value the active managers bring into the investment process.Our outcomes show evidence of the IVOL Puzzle in the post-crisis period[7].

4.2 Heterogeneity of investors' investment horizons hypothesis

Second goal of the thesis is to test the heterogeneity of investors' investment horizon hypothesis, that started to be developed from Brandt et al. (2009). Key intuition behind is the presence of several kind of investors, which implies heterogeneity of their needs and consequently of their investment horizons in financial markets. Our analysis, till now, doesn't consider the single time scales contribution as Malagon et al. (2015) and Yin et al. (2019), it evaluates instead the performance of different degrees of activeness in the portfolio management. However, the two approaches are not completely separated. In fact, if increasing the holding period the IVOL Puzzle weakens, this could mean that investors with a bigger time scale are demanding a premium for bearing idiosyncratic risk.

Since a given holding period return should be the results of all the investment horizons that compose the period, the fact we spot a different compensation for a different holding period is an evidence supporting the heterogeneity of investors' investment horizon hypothesis.

[7] 2010-2018.

Generally speaking, our results show that the sign compensation for bearing idiosyncratic risk is holding-period dependent following an inverted U-shape trend.

4.3 Cumulative Returns

Plotting the cumulative returns some famous strategies (Figure 1), we examine how much investing in each strategy since January 2010 yields for each month over time. We observe that P1 strategy outperforms every other strategy in terms of mean return, Sharpe Ratio and cumulative return. P5-P1 strategy slightly underperforms the momentum strategy in terms of cumulative returns. P5-P1 is a long-short equity strategy, this technique is often used by hedge funds to gain both from the increase and decrease of prices of different securities in the market. A portfolio with this setting protects itself from losses during market downturns, because of this when the strategy has a close to zero correlation relative to the market it's called a "market-neutral" strategy. The beta β of this strategy is -0.56 therefore the strategy, besides performing on average better than Size and Value strategies, can be used as a hedging instrument against market risk. Size and Value strategies as expected from summary statistics we computed, perform poorly post-crisis.

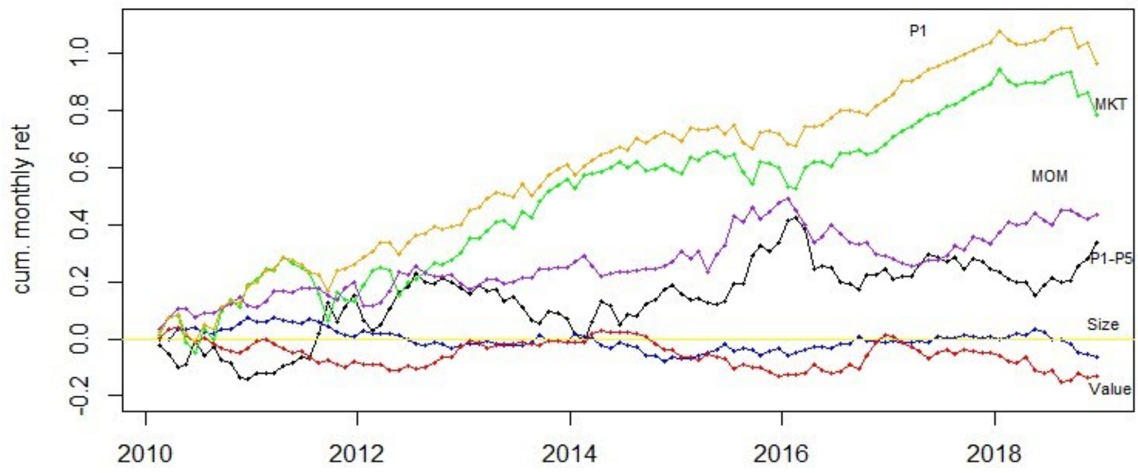

Figure 1: Cumulative returns for each strategy: P1(yellow), Market(green), Momentum(purple), P1-P5(black), Size(blue) and Value(red). Sample period: January 2010-December 2018.

5. Fourier and Wavelet methods for time series in Finance

5.1 From time domain to frequency domain

In several fields, the time domain analysis of a variable can be enhanced by the frequency domain analysis. A time domain series is a variable which is function of time, therefore is indexed in time order and plotting the variable we obtain a time amplitude representation. Studying the frequencies of a process, we can observe characteristics hidden in the frequency domain representation. All the frequency components of a signal/process/series are called frequency spectrum. The mathematical tools used to go from time to frequency domain are generally called transforms and the most popular one is the Fourier transform.

5.2 Wavelet transform

Fourier transform requires the process to be stationary, because it goes from the time-domain to the frequency domain. In finance often the data doesn't satisfy the stationarity requirement. A recent transform which overcome this issue is the wavelet transform because with wavelets we can obtain a time-frequency representation of our data.

5.3 The Discrete Wavelet Transform (DWT)

Contrary to CWT, the Discrete Wavelet Transform has a limited amount of coefficients because the *mother wavelet* is dilated and translet a limited number of times. This is obtained setting:

$$s = 2^{-j} \qquad u = k2^{-j} \tag{2}$$

where j,k are the set of discrete translation and dilatation, implying that the wavelet transform is calculated only at *dyadic* scales (2^j). Another implication is that, being N the observations of our time series, the largest number of scales is the the integer J:

$$J = [log_2(N)] = [log(N)/log(2)] \tag{3}$$

this can be an issue because if the time series is not of *dyadic* length, observations must be added or removed. Two discrete wavelet filter are behind the DWT. One is the *mother wavelet*, denoted $h_l = (h_o, ..., h_{L-1})$. The second one is the *father wavelet*, denoted $g_l = (g_0, ...,g_{L-1})$. Properties of the *mother wavelet* are:

$$\sum_{L=0}^{L-1} h_l = 0, \qquad \sum_{L=0}^{L-1} h_l^2 = 1, \qquad \sum_{L=0}^{L-1} h_l h_{l+2n} = 0 \qquad \forall\, n \in \mathbb{N}_0 \tag{4}$$

Thanks to the above properties, h_l is a difference operator, the DWT has the variance of the original data and a multiresolution analysis can be performed. The *father wavelet* is a low pass filter and captures the long scales, hence the low frequency, smooth components of the series computing the "scaling" coefficients. The *mother wavelet* is a high pass filter and captures the short scales, high frequency, details components of the series. The *father wavelet* has the following condition:

$$\sum_{L=0}^{L-1} g_l = 1 \qquad (5)$$

The first level of decomposition computes the wavelet and the scaling coefficients of the first scale, respectively $w_1(t)$ and $v_1(t)$ that are obtained in the following way:

$$w_1(t) = \sum_{L=0}^{L-1} h_l x(t') \quad and \quad v_1(t) = \sum_{L=0}^{L-1} g_l x(t') \qquad (6)$$

in which $t = 0, 1, ..., T/2 - 1$ and $t' = 2t + 1 - lmodT$.

Thanks to the pyramid algorithm (procedure in figure 2), we can further decompose the low frequency scaling coefficients $v_1(t)$ into other two components. Therefore the second level decomposition has $w = [w_1, w_2, v_2]$ and the J level decomposition has $w = [w_1, ..., w_J, v_J]$.

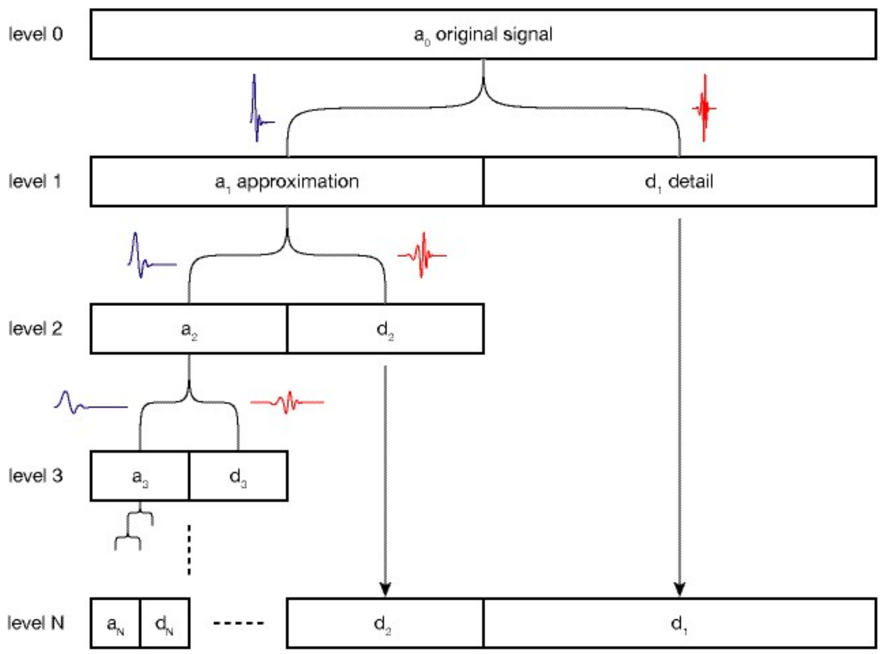

Figure 2: Pyramid algorithm doing a N level decomposition from Sundling et al. (2006).

5.4 The Maximal Overlap Discrete Wavelet Transform (MODWT)

To overcome the limitations we briefly described of DWT, we can use the MODWT. Contrary to the DWT, MODWT consider all the possible (integer) translations. Hence for every scale the wavelet coefficients, the scaling coefficients and the original series have the same length. At the first level decomposition with

MODWT we have:

$$w_1(t) = \sum_{L=0}^{L-1} h_l x(t') \quad and \quad v_1(t) = \sum_{L=0}^{L-1} g_l x(t')$$

in which $t = 0, 1, ..., T$ and $t' = t - lmodT$.

Using the pyramid algorithm, we can obtain the MODWT coefficients for further level of decomposition.

6. Heterogeneity of investors' investment horizons hypothesis and Wavelet transform

Comparing the compensation required by investors, with different holding period, to bear idiosyncratic risk we observe for short holding period (1-3 months) a negative premium, for medium holding period (3-6 months) a zero premium and for long term holding period (6-12 months) again a negative compensation. Our hypothesis is that this result can be driven by investors with different investment horizon requiring different compensation for bearing idiosyncratic risk. Testing the trading strategy with increasing holding period can show the compensation of all the investors with investment horizon smaller or equal to the holding period but can't properly disentangle every required compensation. Therefore the return we observe at the end of the holding period of p months, is the aggregation of all the compensation required by investment horizons smaller or equal to p.

To study in detail the Idiosyncratic Volatility Puzzle, we use the Wavelet transform to study the contribution of each frequency (time scale) to the final holding period return of portfolio 5 and 1.

6.1 From frequencies to investors

We summarized in Section 9 the theoretical background, the properties and the applications of the wavelet transform. In finance, decomposing in frequencies a time series of returns, we look for an economical interpretation of the frequencies. Considering a short (long) time scale of our time series, means to capture the high (low) frequency contributions to our series. Therefore in our case, the high frequencies (short time scale) are the contribution of the short term investors to the series, while the low frequencies (long time scale) represent the contribution of the long term investors. This reasoning is reasonable as the short (long) term investors contribute to the most (least) frequent movements of the price. Different investors have different trading frequencies (Malagon et al. (2015))

The definition of short/medium/long term investors depends by the frequency of the data on which we do the wavelet transform. The technique creates frequency bands separated by multiples of 2^J. Therefore with daily data we can capture the contribution of investors with investment horizon of 2-4 days, 4-8 days, 8-16 days till the maximum admitted level of decomposition.

Performing the wavelet transform, we decompose our time series S_0 into an approximation S_j (long time scale/low frequency) and details D_j (short time scale /high frequency).

6.2 Application to trading strategy 1/0/1

We apply a wavelet transform of level 6 to our framework, decomposing the daily returns and the daily factors into 6 details and one smooth component (example in figure 3). We use the la8 filter and the Maximal Overlap Discrete Wavelet Transform. The reason behind the MODWT is that we need to apply the transform to approximately 11.000 firms. To avoid issues linked to the required *dyadic* length of the data by the Discrete Wavelet Transform, we choose the MODWT. Once the data is decomposed, we test the trading strategy 1/0/1 for each details and smooth components. Aim of this framework is to capture the compensation required, for bearing idiosyncratic risk, by specific investors' investment horizon identified by each of the details and smooth components. The next sections cover just detail 1, 5 and 6 while the others are in the appendix.

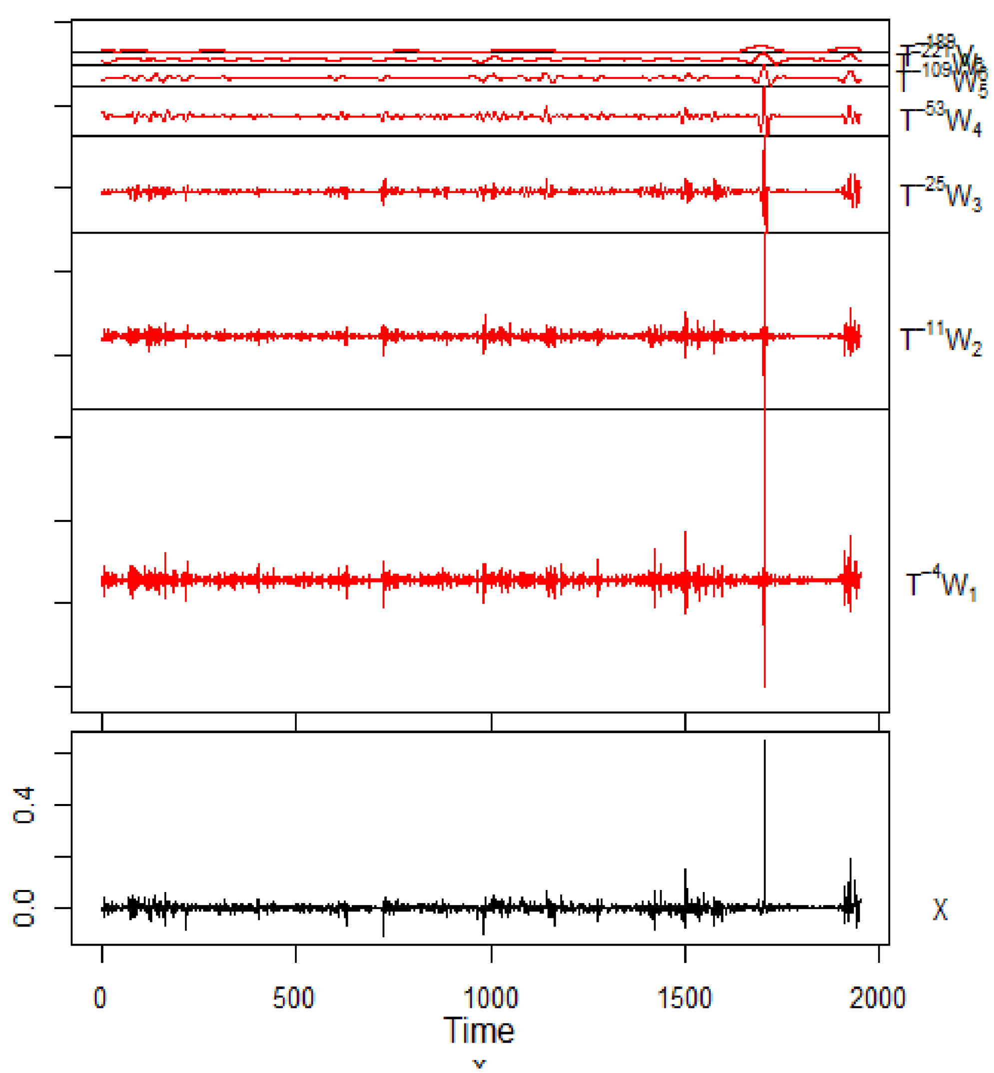

Figure 3: X is the time series of daily returns of an American firm in our dataset. Applying the wavelet transform with level 6, we obtain six details component called W and one smooth component defined V.

6.3 D1, D5, D6 and S6 component

The D1 component, since the wavelet transform is done on daily data, represents the investment horizon from 2 to 4 days. Results are in Table 4.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	-0.06	0.40	0.23	-0.08***	-0.08***
				[-2.79]	[-3.06]
P2	-0.10	0.44	0.38	-0.12***	-0.12***
				[-4.13]	[-3.97]
P3	-0.15	0.51	0.24	-0.16***	-0.16***
				[-4.29]	[-4.51]
P4	-0.24	0.58	0.12	-0.26***	-0.26***
				[-7.45]	[-7.30]
P5	-0.42	0.70	0.04	-0.43***	-0.43***
				[-9.54]	[-9.96]
P5-P1	-0.35**			-0.34***	-0.34***
	[-12.4]			[-12.54]	[-12.67]

Table 4: D1 component, 2-4 days investment horizon. Forming value-weighted quintile portfolios month we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in monthly returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) *t*-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/1.

The trading strategy P5-P1, which represents the compensation for exposure to idiosyncratic risk, is negative and statistically significant. This result is in line with our previous results because we find strategy 1/0/1 (Table 1) having a negative monthly compensation. Alphas relative to CAPM and Fama-French 3-factor model are negative and statistically significant. We find the same patterns for standard deviation market share observed in Table 1.

6.4 D5 component

The D5 component, represents the investment horizon from 32 to 64 days. Running the analysis we find the results in Table 5.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	-0.03	1.40	0.19	-0.23***	-0.24***
				[-3.02]	[-3.03]
P2	-0.01	1.52	0.28	-0.21**	-0.23***
				[-2.21]	[-2.78]
P3	-0.01	1.59	0.26	-0.23***	-0.25***
				[-3.00]	[-3.34]
P4	-0.07	1.75	0.18	-0.32***	-0.34***
				[-3.59]	[-4.05]
P5	0.04	2.08	0.08	-0.22**	-0.24**
				[-2.15]	[-2.29]
P5-P1	0.07			0.01	0.01
	[1.22]			[0.08]	[0.04]

Table 5: D5 component, 32-64 days investment horizon. Forming value-weighted quintile portfolios month we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in monthly returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French threefactor models. Robust Newey and West (1986) *t*-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/1

The trading strategy P5-P1, which represents the compensation for exposure to idiosyncratic risk, is positive. Alphas relative to CAPM and Fama-French 3-factor model are positive too. Table 5 shows investors with time horizon from 32 to 64 days demand a positive compensation for bearing idiosyncratic risk.

6.5 D6 component

The D6 component, represents the investment horizon from 64 to 128 days. In Table 6 there are our findings.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	-0.01	1.18	0.21	-0.15*	-0.15*
				[-1.75]	[-1.80]
P2	-0.02	1.46	0.28	-0.21***	-0.20***
				[-2.62]	[-2.66]
P3	0.02	1.56	0.26	-0.18**	-0.18*
				[-1.96]	[-1.90]
P4	0.07	1.92	0.18	-0.19*	-0.17
				[-1.70]	[-1.58]
P5	0.09	2.23	0.09	-0.19	-0.17
				[-1.53]	[-1.34]
P5-P1	0.10			-0.04	-0.02
	[1.51]			[-0.31]	[-0.15]

Table 6: D6 component, 64-128 days investment horizon. Forming value-weighted quintile portfolios month we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in monthly returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) *t*-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/1

The trading strategy P5-P1, which represents the compensation for exposure to idiosyncratic risk, is positive. Alphas relative to CAPM and Fama-French 3-factor model are positive too. Table 6 shows investors with time horizon from 64 to 128 days demand a positive compensation for bearing idiosyncratic risk.

6.6 The smooth component

The smooth component, represents the time scales longer than 128 days. Results are in Table 7.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	1.26	1.70	0.14	1.07***	1.10***
				[4.20]	[4.66]
P2	1.11	1.80	0.23	0.92***	0.96***
				[4.00]	[4.00]
P3	1.05	1.59	0.27	0.88***	0.90***
				[4.11]	[4.29]
P4	1.14	1.76	0.24	0.93***	0.97***
				[3.92]	[3.70]
P5	1.19	2.31	0.12	0.94***	0.99***
				[2.75]	[3.23]
P5-P1	-0.06			-0.13	-0.11
	[-0.47]			[-1.01]	[-0.85]

Table 7: Smooth component, > 128 days investment horizon. Forming value-weighted quintile portfolios month we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in monthly returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French three-factor models. Robust Newey and West (1986) *t*-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/1

The compensation to bear idiosyncratic risk is negative again. The inverted U-shape the compensation draws for increasing time scales further explains our results in Section 4. The not constant performance of portfolio with increasing holding and rebalancing period, finds justification in the heterogeneity contribution of different frequencies to the series.

6.7 Wavelet transform's results

In Section 3 we interpret the performance of a trading strategy based on idiosyncratic volatility with increasing holding period. Referring to the new hypothesis in literature about the Puzzle being driven by the heterogeneity of investors' investment horizons hypothesis, we state that the almost null performance of trading strategy 1/0/6 could be motivated by a positive compensation demanded by investors with investment horizon between the 3rd and 6th month. This hypothesis finds support in section 6 where we apply the wavelet transform to study the different frequencies in our data. We observe a negative compensation for time scales going from 2 to 32 days, while the the investors with time scales from 32 to 128 days require a positive compensation. The smooth component, representing the long run gets negative again. Since in our data months are approximately 20 days long, we have a positive compensation in time scales from 1 month and a half to 6 months. The decomposition we perform brings additional evidence about the relevance of the heterogeneity of investors' investment horizons hypothesis. Our setting for the wavelet transform follows Malagon et al. (2015), who work with daily data and performs the Ang et al. (2006) trading strategy. Malagon et al. (2015) finds a negative compensation for short term investors and a positive one for the smooth component but fails to find the time scale where the compensation gets negative again. Yin et al. (2019) instead, working with monthly data and the Common Idiosyncratic Volatility factor (CIV, derived by principal component analysis), finds an inverted U-shape in the factor loadings of CIV. Because we observe an inverted U-shape too for the compensation for bearing idiosyncratic risk, our findings are in line with Yin et al. (2019).

7. Conclusion

Our study uses Ang et al. (2006)'s framework to examine if the Idiosyncratic Volatility Puzzle post-crisis (2010-2018) is still present. Additionally, to analyse if the relationship between idiosyncratic volatility and returns is holding-period dependent we apply the following trading strategies: 2/0/1, 1/0/1, 1/0/3, 1/0/6 and 1/0/12. The strategies are constructed applying the L/M/N framework changing the parameters to test different holding periods. Our study shows a negative relationship between idiosyncratic volatility and returns for trading strategy 1/0/1. We compute several performance evaluation measures P1-P5, P1, Market, Size, Value and Momentum strategies. We find P1 (low idiosyncratic volatility stocks) outperforms every other strategy in reward-risk terms. Moreover P1-P5 strategy can be used for hedging purposes because of its long-short equity structure and its negative correlation with the market. Moreover, we discover that the cost of ignoring the low idiosyncratic volatility cost for a mean-variance optimizer investor, holding an indexed allocation, is a sizeable increase of its utility function.

Second goal of the thesis is to test if the Puzzle is holding-period dependent. Studying the L/M/N strategy with different N (and different rebalancing periods), we bring some evidence in line with the heterogeneity of investors' investment horizons hypothesis. All the strategies but 1/0/6 present a negative compensation for bearing idiosyncratic risk. Trading strategy 1/0/6 shows no presence of IVOL Puzzle, because there is no difference in performance between portfolios made of high IVOL stocks and portfolios made of low IVOL stocks. The absence of a compensation for bearing firm-specific risk means that investors with investment horizon from 4 to 6 months (at least) start to require a positive compensation to bear idiosyncratic risk, making the final compensation required close to zero at the end of 6th month.

Overall the findings of this thesis prove the presence of the IVOL Puzzle post-crisis and bring evidence in line with heterogeneity of investors' investment horizons hypothesis supported by Malagon et al. (2015) and Yin et al. (2019).

Thanks to the decomposition of our data provided by the wavelet transform, we prove as the almost null performance of strategy 1/0/6 is driven by a positive compensation for bearing idiosyncratic risk demanded in time scales preceding the 6th month. Moreover this technique furnishes additional evidence to the importance of the heterogeneity of investors' investment horizons hypothesis, increasing the accuracy of the analysis and quality of the model. The heterogeneity of investors' investment horizon hypothesis can be one of the reasons behind the mixed literature. The several approaches to the topic, the different equity markets studied and the different time period considered have contributed to make the construction of a proper consensus over the IVOL Puzzle difficult. Nevertheless, besides being reasonable the heterogeneity of investors' investment horizon hypothesis seems to put together different results instead of generating additional stances.

8. Limitations and recommendations for future research

The Idiosyncratic Puzzle has been investigated with several approaches in literature. Interesting would be to search for evidence supporting our results trying to estimate expected IVOL as Fu (2009) instead of using the 1 month lagged IVOL as a proxy. Additional evidence in line with our finding coming from a different framework would bring robustness to the study.

A limitation regards how to handle properly the delisting firms over our time series. Some firms enter into the dataset after 2010 and some disappear before 2018. When a firm's time series of returns stops, that firm has been delisted. Delisting is usually a bad sign but it's not always the case, for example delisting can be a firm's choice or due to a merger. The limitation in our case is that delisted stocks, especially when we compute trading strategy with holding period bigger than 1 month, were simply not considered in the next period portfolio creating a loss of information. A method to handle this could improve the accuracy of the results.

A more general limitation of our study is the lack of robustness check. Ang et al. (2006) checked their results for size, book-to-market, leverage, liquidity, volume, turnover, bid-ask spread, coskewness and dispersion of analysts' forecasts. In addition Ang et al. (2009) tested their results for: market frictions, information dissemination and option pricing.

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APPENDIX

D2 component

The D2 component, represents the investment horizon from 4 to 8 days. Running the analysis, we obtain Table 8.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	-0.01	0.39	0.22	-0.04*	-0.05**
				[-1.81]	[-2.30]
P2	-0.03	0.47	0.37	-0.06**	-0.07***
				[-2.33]	[-2.79]
P3	-0.06	0.53	0.24	-0.10***	-0.10***
				[-3.26]	[-3.92]
P4	-0.07	0.60	0.13	-0.11***	-0.12***
				[-3.15]	[-3.95]
P5	-0.15	0.68	0.05	-0.19***	-0.20***
				[-7.20]	[-6.20]
P5-P1	-0.14**			-0.15***	-0.16***
	[-6.62]			[-7.80]	[-7.50]

Table 8: D2 component, 4 to 8 days investment horizon. Forming value-weighted quintile portfolios month we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in monthly returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French threefactor models. Robust Newey and West (1986) *t*-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/6.

The trading strategy P5-P1, which represents the compensation for exposure to idiosyncratic risk, is almost zero. This result is in line with our results because we find strategy 1/0/1 having a negative monthly compensation. Alphas relative to CAPM and Fama-French 3-factor model are negative.

D3 component

The D3 component, since the wavelet transform is done on daily data, represents the investment horizon from 8 to 16 days. Results in Table 9.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	-0.03	0.55	0.21	-0.09***	-0.10***
				[-3.20]	[-3.75]
P2	-0.01	0.64	0.34	-0.07***	-0.08***
				[-2.57]	[-2.80]
P3	-0.03	0.77	0.25	-0.11***	-0.12***
				[-3.30]	[-4.42]
P4	-0.03	0.85	0.14	-0.12***	-0.13***
				[-2.73]	[-3.18]
P5	-0.05	0.96	0.05	-0.14***	-0.15***
				[-2.68]	[-3.54]
P5-P1	-0.02			-0.05	-0.06
	[-0.86]			[-1.43]	[-1.40]

Table 9: D3 component, 8-16 days investment horizon. Forming value-weighted quintile portfolios month we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in monthly returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French threefactor models. Robust Newey and West (1986) *t*-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/1.

The trading strategy P5-P1, which represents the compensation for exposure to idiosyncratic risk, is almost zero. This result is in line with our results because we find strategy 1/0/1 having a negative monthly compensation. Alphas relative to CAPM and Fama-French 3-factor model are negative.

D4 component

The D4 component, since the wavelet transform is done on daily data, represents the investment horizon from 16-32 to 4 days. Results in Table 10.

Rank	Mean	Std. Dev.	MKT Share	CAPM alpha	FF3 alpha
P1	0.96	2.78	0.20	0.46	0.40
				[2.84]	[3.12]
P2	1.00	3.47	0.37	0.36	0.32
				[2.74]	[2.80]
P3	0.87	4.37	0.26	0.05	0.05
				[0.46]	[0.40]
P4	0.77	4.99	0.13	-0.15	-0.11
				[-0.94]	[-0.73]
P5	0.51	5.64	0.05	-0.48	-0.40
				[-2.04]	[-1.86]
P5-P1	-0.45			-0.94	-0.80
	[-1.05]			[-2.86]	[-3.05]

Table 10: D4 component, 16-32 days investment horizon. Forming value-weighted quintile portfolios month we sort stocks based on idiosyncratic volatility relative to Fama and French (1993). Volatility is computed using daily data from the previous month. P1 (P5) is the portfolio with the lowest (highest) idiosyncratic volatilities. The statistics Mean and Std. Dev. are measured in monthly percentage terms over (not excess) simple returns. MKT Share is the average relative MKT share of the portfolio. P5-P1 refers to the difference in monthly returns between Portolio 5 and 1. The last two columns are the Jensen's alphas relative to CAPM and Fama-French threefactor models. Robust Newey and West (1986) *t*-statistics are reported in the square brackets. *** means the value is statistically significant at 1% level, ** at 5% level and * at 10% level from a two-tailed test. Sample period is 2010-2018, trading strategy is 1/0/1.

The trading strategy P5-P1, which represents the compensation for exposure to idiosyncratic risk, is negative. This result is in line with our results because we find strategy 1/0/1 having a negative monthly compensation. Alphas relative to CAPM and Fama-French 3-factor model are negative.